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Research on the Optimal Investment Strategy with Jumps When Risks Cannot Be Hedged

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Abstract: The problem of the optimal investment strategy has always been a key research content in modern finance. Since stock prices in real financial markets often experience jumps, and the randomness of investors' labor income contributes to risks that cannot be completely hedged, it is necessary to consider these factors in investment strategy design. This paper studies the continuous-time dynamic mean-variance portfolio selection problem when the risk is not hedged. It is assumed that the price of risky assets follows a jump-diffusion process. The investor's goal is to minimize the variance of the wealth at the terminal time under the condition of a given expected terminal wealth. By solving the corresponding Hamilton-Jacobi-Bellman equation of the model, the viscosity solution of the optimal investment strategy is obtained. The results show that the jump factors in the price process and the unhedged risk have an impact on the optimal investment strategy that cannot be ignored.

Keywords: jump-diffusion model; optimal investment portfolio; HJB equation; viscous solution

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1. Introduction

In 1952, an American economist, innovatively proposed the concept of "portfolio", financial mathematics began to develop rapidly, and has produced many research results under mean-variance models based on various risk measures [1]. At the same time, it has introduced numerous valuable tools in financial theory, such as risk appetite and utility functions. In the field of modern portfolio research, the optimal portfolio is always a very important problem. The optimal portfolio problem refers to the reasonable allocation of risk-free assets and risky assets, so as to maximize the expected utility of investors' wealth at the terminal moment.

Merton made a pioneering and systematic study on how to maximize the expected utility at the terminal moment by optimizing investment strategies under continuous time frame [2]. This problem is called the classic Merton problem, and discusses the optimal asset allocation under the additional assumption of investors with fixed relative risk aversion or absolute risk aversion utility function. Fleming and Rishel used Markov dynamic programming method to solve the model and get the analytical solution of the optimal decision [3]. On the basis of this, the paper gives the solution of the optimal investment strategy problem by using martingale and convex duality theory for more general pricing process under perfect market conditions. This solution applies when the constraint condition of the utility function selected by the investor is monotonically increasing and strictly concave. Longo and Mainini studied the portfolio optimization problem of HARA (hyperbolic absolute risk-averse utility) investors [4]. By comparing the optimal portfolio under partial observation with the corresponding short-sighted strategy, they proved that for the situation where the risk is constant, The ratio between a portfolio under partial observation and its shortsighted portfolio increases as risk tolerance increases.

This paper also studies the dynamic mean-variance portfolio selection problem, Because of the existence of unhedged risk, the martingale method cannot be directly applied. In this paper, the optimal strategy is obtained by using the HJB method and the concept of viscous solution.

Section 2 develops a dynamic model of asset prices with jumps when risks are unhedged, in which risk asset prices follow the jump-diffusion process. Optimization problems are formulated with the goal of minimizing the risk of terminal wealth under a given expected value. Section 3 solves the optimal investment strategy by HJB equation and viscous solution. Section 4 summarizes the main findings of this paper.

2. Model Construction: Asset Assumption

This section assumes the investor's investment objective is to maximize the wealth at the end of the investment term with a certain risk tolerance. The price process of risky securities at the time of t follows the stochastic differential equation with jumps:

$$dS(t) = S(t^-)\{\mu(t)dt + \sigma(t)dW_1(t) + \nu(t)dN(t)\}, t \in [0, T]$$

The investor's random labor income is subject to a random differential equation:

$$dy(t) = y(t)\{a(t)dt + b(t)dW_2(t)\}, t \in [0, T]$$

Assume that the risk-free rate is $r(t), r(t), \mu(t), \sigma(t), a(t), b(t)$ are deterministic, Borel measurable, positive, bounded functions on $[0, T]$. $N(t)$ is a Poisson process for $\{\mathcal{F}_t\}_{t\geq 0}$ adaptation on a probability space $(\Omega, \mathcal{F}, P)$ with a strength of λ. $W_1(t)$ and $W_2(t)$ are $\{\mathcal{F}_t\}_{t\geq 0}$ adaptive Brownian motion on probability space $(\Omega, \mathcal{F}, P)$. The correlation coefficient between the two is ρ.

Both $W_1(t)$ and $W_2(t)$ are independent of $N(t)$. $\pi(t)X(t)$ is a wealth invested in risky assets at time t. One strategy $\pi(t)$ is feasible. If it is $\mathcal{F}_t$ sequentially measurable, and at any time $t \geq 0$ it is satisfied $E\left[\int_0^T \pi^2(t)X^2(t)dt\right] < +\infty$. At the same time, because investors are rational, excessive jump in risk assets may lead to the bankruptcy of investors, so investors are not allowed to short risk assets, that is, they must be satisfied $\pi(t) \geq 0$. Denote all policy feasible sets as Π. So, the investor's wealth process $X(t)$ satisfies the stochastic differential equation:

$$\begin{aligned} dX(t) = X(t)rdt + (\mu(t) - r)\pi(t)X(t)dt + y(t)dt + \sigma(t)\pi(t)X(t)dW_1(t) \\ + \nu(t)\pi(t)X(t)dN(t) \end{aligned}$$

The goal of the investor is to find a feasible strategy in the set of feasible strategies where the end-time wealth meets $E\big(X(T)\big) = d$, while minimizing the risk of terminal wealth: $Var[X(T)] = E[X(T) - EX(T)]^2 = E[X(T) - d]^2$, Therefore, the above problems can be expressed as optimization problems:

$$minVar[X(T)] = E[X(T) - d]^2$$

$$s.t. \begin{cases} E[X(T)] = d, \\ \pi(\cdot) \in \Pi \end{cases}$$

This convex optimization problem can be solved by introducing the Lagrange factor $\alpha \in R$. According to the reference, the problem is equivalent to the following problem [1]:

$$\min \ E[X(T) - k]^2$$

Here $k = d - \alpha$, just ask for the optimal solution of the auxiliary problem, and then use the Lagrange duality theorem to get the optimal strategy and the optimal value function of the original problem, so here only solve the optimal solution of the auxiliary problem, first, define the optimal value function:

$$V(t, x, y) = \min_{\pi \in \Pi} [E(X(T) - k)^2 | X(t) = x]$$

Since the solution of the HJB equation corresponding to the above random problem does not have the required smoothness conditions, the concept of viscous solution must be used [5]. The following HJB equation is studied:

$$\min_{\pi\geq 0}\{V_t + V_x[r(t)x + (\mu(t) - r(t))\pi x + y] + \frac{1}{2}V_{xx}\sigma^2(t)\pi(t)^2x^2$$
$$+\frac{1}{2}V_{yy}a^2(t) + V_y b(t) + V_{xy}\rho a(t)\sigma(t)\pi(t)x$$
$$+\lambda E[V(t, x + v\pi, y) - V(t, x, y)]\} = 0.$$

The boundary conditions are:

$$V(T, x, y) = (x - k)^2$$

Here the optimally valued function $V(t, x, y)$ is not twice continuously differentiable. So the concept of viscous solution is used to study the optimization problem. Now, the concept of viscous solution is given.

Definition 2.1:

1) The continuous function v defined on $(t, x, y) \in [0, T] \times R \times R$ is the viscosity subsolution of the above equation, if for any quadratic continuously differentiable function $\varphi: [0, T] \times R \times R$, At any point $(\bar{t}, \bar{x}, \bar{y}) \in [0, T] \times R \times R$ is satisfied:

$$\min_{\pi\geq 0}\{\varphi_t(\bar{t}, \bar{x}, \bar{y}) + \varphi_x[r(\bar{t})\bar{x} + (\mu(\bar{t}) - r(\bar{t}))\pi\bar{x} + \bar{y}] + \frac{1}{2}\varphi_{xx}\sigma^2(t)\pi^2\bar{x}^2 + \frac{1}{2}\varphi_{yy}a^2(t) + \varphi_y b(t) + \varphi_{xy}\rho a(t)\sigma(t)\pi\bar{x} + \lambda E[\varphi(\bar{t}, \bar{x} + v\pi, \bar{y}) - \varphi(\bar{t}, \bar{x}, \bar{y})]\} \geq 0$$

Where $(\bar{t}, \bar{x}, \bar{y})$ is the maximum point of the function $v - \varphi$ defined on $[0, T] \times R \times R$, and $v(\bar{t}, \bar{x}, \bar{y}) = \varphi(\bar{t}, \bar{x}, \bar{y})$.

2) The continuous function v defined on $(t, x, y) \in [0, T] \times R \times R$ is the viscosity supersolution of the above equation, if for any quadratic continuously differentiable function $\varphi: [0, T] \times R \times R$, At any point $(\bar{t}, \bar{x}, \bar{y}) \in [0, T] \times R \times R$ is satisfied:

$$\min_{\pi\geq 0}\{\varphi_t(\bar{t}, \bar{x}, \bar{y}) + \varphi_x[r(\bar{t})\bar{x} + (\mu(\bar{t}) - r(\bar{t}))\pi\bar{x} + \bar{y}] + \frac{1}{2}\varphi_{xx}\sigma^2(t)\pi^2\bar{x}^2 + \frac{1}{2}\varphi_{yy}a^2(t) + \varphi_y b(t) + \varphi_{xy}\rho a(t)\sigma(t)\pi\bar{x} + \lambda E[\varphi(\bar{t}, \bar{x} + v\pi, \bar{y}) - \varphi(\bar{t}, \bar{x}, \bar{y})]\} \leq 0$$

3) The function v defined on $(t, x, y) \in [0, T] \times R \times R$ is the viscous solution of the equations if it is both the viscosity subsolution and the viscosity super solution. Symbols are introduced for convenience:

$$\Phi(t) = [\sigma^2(t) + \lambda v(t)]x^2, B(t) = [\mu(t) - r(t) + \lambda v(t)]x$$
$$\phi(t) = \frac{B^2(t)}{\Phi(t)}, P_1(t) = e^{\int_t^T \phi(s)-2r(s)ds}, \alpha_1(t) = Q_1(t)y - \frac{Q_1^2(t)}{4P_1(t)}\phi(t)$$
$$Q_1(t) = e^{\int_t^T \phi(s)-r(s)ds}\left[-\int_t^T 2ye^{\int_s^T -r(u)du}ds - 2k\right]$$
$$R_1(t, y) = [k + \int_t^T \alpha_1(s)e^{-r(s)(s-t)}ds]^2, P_2(t) = e^{-2\int_t^T r(s)ds}$$
$$Q_2(t) = e^{\int_t^T -r(s)ds}[-\int_t^T 2ye^{\int_s^T -r(u)du}ds - 2k], R_2(t, y) = [k + \int_t^T Q(s)ye^{-r(s)(s-t)}ds]^2$$
$$\frac{Q_1(t)}{2P_1(t)} = \frac{Q_2(t)}{2P_2(t)} = -e^{\int_t^T r(s)ds}\left[\int_t^T ye^{\int_s^T -r(u)du}ds + k\right]$$

3. Model Solution

3.1. Theorem 1

$$v(t, x, y) = \begin{cases} P_1(t)x^2 + Q_1(t)x + R_1(t, y), x - e^{\int_t^T r(s)ds}\left[\int_t^T ye^{\int_s^T -r(u)du}ds + k\right] < 0 \\ P_2(t)x^2 + Q_2(t)x + R_2(t, y), x - e^{\int_t^T r(s)ds}\left[\int_t^T ye^{\int_s^T -r(u)du}ds + k\right] \geq 0 \end{cases}$$

It is the viscous solution of HJB equation, and the optimal investment strategy is:

$$\pi^*(t,x,y)=\begin{cases}-\left\{x-e^{\int_t^T r(s)ds}\left[\int_t^T ye^{\int_s^T -r(u)du}ds+k\right]\right\}\dfrac{B(t)}{\Phi(t)}, \\ x-e^{\int_t^T r(s)ds}\left[\int_t^T ye^{\int_s^T -r(u)du}ds+k\right]<0 \\ 0, x-e^{\int_t^T r(s)ds}\left[\int_t^T ye^{\int_s^T -r(u)du}ds+k\right]\geq 0\end{cases}$$

Proof. Guess that $v(t,x,y)=P(t)x^2+Q(t)x+R(t,y).P(t),Q(t),R(t,y)$ are adaptation functions, assuming$\forall t\in[0,T],P(t)>0$ was established. According to the boundary conditions, we can obtain $P(T)=1,Q(T)=-2k,R(T,y)=k^2$, the above form generation can get into the HJB equation:

$$2P(t)\min_{\pi\geq 0}\left[\frac{1}{2}\Phi(t)\pi^2+\left(x+\frac{Q(t)}{2P(t)}\right)B(t)\pi\right]+$$
$$[2P(t)x+Q(t)](r(t)x+y)+\frac{1}{2}R_{yy}a^2(t)+R_y b+P_t x^2+Q_t x+R_t=0$$

Since $P(t)>0$, when the investment strategy $\pi(t)$ is unconstrained, the above formula reaches its minimum at the following points:

$$\pi^0(t,x,y)=-\left(x+\frac{Q(t)}{2P(t)}\right)\frac{B(t)}{\Phi(t)}$$

If the right side of the above formula is less than 0, it is truncated with a zero value. Therefore, the following areas are defined:

$$A_1=\{(t,x,y)\in[0,T]\times R\times R,\pi^0>0\}$$
$$A_2=\{(t,x,y)\in[0,T]\times R\times R,\pi^0\leq 0\}$$

First analyze the situation in the region A_1. At this point, the minimum value π^0 is substituted into the above formula, which can be obtained:

$$[P_t(t)+2r(t)P(t)-\phi(t)P(t)]x^2+[2P(t)y+Q(t)r(t)+Q_t(t)-\phi(t)Q(t)]x+$$
$$\left[Q(t)y+R_y b+R_t+\frac{1}{2}R_{yy}a^2(t)-\frac{Q^2(t)}{4P(t)}\phi(t)\right]=0$$

Using the comparison coefficient method, it can be inferred that $P(t),Q(t),R(t)$ satisfy the differential equation:

$$\begin{cases}P_t(t)+2r(t)P(t)-\phi(t)P(t)=0, \\ P(T)=1,\end{cases}$$
$$\begin{cases}2P(t)y+Q(t)r(t)+Q_t(t)-\phi(t)Q(t)=0, \\ Q(T)=-2k,\end{cases}$$
$$\begin{cases}Q(t)y+R_y b+R_t+\dfrac{1}{2}R_{yy}a^2(t)-\dfrac{Q^2(t)}{4P(t)}\phi(t)=0, \\ R(T)=k^2,\end{cases}$$

By solving the above three equations, we can get:

$$v(t,x,y)=P_1(t)x^2+Q_1(t)x+R_1(t,y)$$

Therefore, $v(t,x,y)$ is the solution of the region A_1, the first part is proved, and the second part is proved below. When $(t,x,y)\in A_2$, the HJB equation can be rewritten as:

$$[P_t(t)+2r(t)P(t)-P(t)]x^2+[2P(t)y+Q(t)r(t)+Q_t(t)]x$$
$$+\left[Q(t)y+R_y b+R_t+\frac{1}{2}R_{yy}a^2(t)\right]=0$$

Using the comparison coefficient method, it can be inferred that $P(t),Q(t),R(t)$ satisfy the differential equation:

$$\begin{cases}P_t(t)+2r(t)P(t)=0, \\ P(T)=1,\end{cases}$$
$$\begin{cases}2P(t)y+Q(t)r(t)+Q_t(t)=0, \\ Q(T)=-2k,\end{cases}$$
$$\begin{cases}Q(t)y+R_y b+R_t+\dfrac{1}{2}R_{yy}a^2(t)=0, \\ R(T)=k^2,\end{cases}$$

By solving the above three equations, we can get:

$$v(t,x,y)=P_2(t)x^2+Q_2(t)x+R_2(t,y)$$

Let's verify that $v(t,x,y)$ is continuously differentiable and define the boundary surface:

$$A_0 = \left\{(t,x,y) \in [0,T] \times R \times R, x - e^{\int_t^T r(s)ds}\left[\int_t^T ye^{\int_s^T -r(u)du}ds + k\right] = 0\right\}$$

It is easy to verify that in $A_1 \cup (A_2 \setminus A_0), v(t,x,y)$ is continuously differentiable, and we will only prove the case of $(t,x,y) \in A_0$. When $(t,x,y) \in A_0$, it is calculated directly to obtain that $P_1(t)x^2 + Q_1(t)x + R_1(t,y) = P_2(t)x^2 + Q_2(t)x + R_2(t,y) = 0, v_{t^-}^{'} = v_{t^+}^{'}, v_{x^-}^{'} = v_{x^+}^{'}, v_{y^-}^{'} = v_{y^+}^{'}$.This shows that for any $(t,x,y) \in A_0, v(t,x,y)$ is also continuously differentiable. But because $P_1(t) \neq P_2(t)$, So in A_0, v_{xx}' does not exist. This shows that $v(t,x,y)$ does not have a sufficiently smooth property to be second-order non-differentiable, and therefore the problem needs to be solved in the framework of viscous solutions. Let $\psi \in C^{1,2}([0,T],R,R), V - \psi$ reaches its maximum value at $A_0. \psi = \psi_t' = \psi_x' = \psi_y' = \psi_{xy}' = \psi_{yy}' = 0$. At the same time, $\psi_{xx}'' \geq 2P_1(t)$. So, in the HJB equation, replacing V with ψ can be obtained:

$$\begin{aligned}
&\min_{\pi\geq 0}\{\psi_t(t,x,y) + \psi_x[r(t)x + (\mu(t) - r(t))\pi x + y] + \frac{1}{2}\psi_{xx}\sigma^2(t)\pi^2x^2 + \frac{1}{2}\psi_{yy}a^2(t) + \\
&\psi_y b(t) + \psi_{xy}\rho a(t)\sigma(t)\pi x + \lambda E[\psi(t,x+\nu\pi,y) - \psi(t,x,y)]\} \\
&= \frac{1}{2}\psi_{xx}\sigma^2(t)\pi^2x^2 + \lambda E[\psi(t,x+\nu\pi,y) - \psi(t,x,y)] \\
&\geq P_1(t)\sigma^2(t)\pi^2x^2 + \lambda E[\psi(t,x+\nu\pi,y) - \psi(t,x,y)] = 0
\end{aligned}$$

Thus $v(t,x,y)$ is a viscosity supersolution of the HJB equation. Similarly, it can be shown that $v(t,x,y)$ is a viscosity subsolution of the HJB equation, so by *Definition 2.1*, $v(t,x,y)$ is a viscous solution of the HJB equation. End Proof.

3.2. Theorem 2

(Verification theorem) $v\,(t,x,y)$ is the viscosity solution of HJB equation. We set $V(t,x,y)$ as value function, for any $(t,x,y) \in [0,T] \times R \times R$, $V(t,x,y) = v(t,x,y)$ are correct.

Proof. Firstly, consider the initial wealth $x - e^{\int_t^T r(s)ds}[\int_t^T ye^{\int_s^T -r(u)du}ds + k] < 0$, for any $\pi \in \Pi$, by Ito's lemma, we can get:

$$\begin{aligned}
&v(T,X_T^\pi,y_T) = v(t,x,y) + \int_t^T \{V_t + V_x[r(s)x + (\mu(s) - r(s))\pi x + y] + \frac{1}{2}V_{xx}\sigma^2(t)\pi(s)^2x^2 \\
&+ \frac{1}{2}V_{yy}a^2(s) + V_y b(s) + V_{xy}\rho a(s)\sigma(s)\pi(s)x + \lambda E[V(t,x+\nu\pi,y) - V(t,x,y)]\}ds \\
&+ \int_t^T \sigma(s)\pi(s)X(s)dW_1(s) + \int_t^T [v(t,x+\nu\pi,y) - v(t,x,y)]d\widetilde{N}(t)
\end{aligned}$$

Compensated Poisson Process $\widetilde{N}(t) = N(t) - \int_0^T \lambda(s)ds$ is a $\mathcal{F}_t$ adapted martingale process, Thus, the above formula can be simplified to:

$$v(T,X_T^\pi,y_T) \geq \quad v(t,x,y) + \int_t^T [v(t,x+\nu\pi,y) - v(t,x,y)]d\widetilde{N}(t)$$

We notice $v(t,x,y) + \int_t^T [v(t,x+\nu\pi,y) - v(t,x,y)]d\widetilde{N}(t)$ is a martingale process, the mathematical expectation is equal to zero, and because $v(T,x,y) = (x-k)^2$, Taking mathematical expectations on both sides of the above expression, we get:

$$[E(X_T^{\pi^*} - k)^2 \mid X(t) = x] \geq v(t,x,y)$$

That means $V(t,x,y) \geq v(t,x,y)$, when the optimal investment strategy $\pi = \pi^*$.The equals sign in the inequality is true: $V(t,x,y) = v(t,x,y)$.We can prove it in the same way for the other case. End Proof [6].

4. Summary

This paper studies a basic issue based on the stochastic optimal control theory. The specific research content is summarized as follows: The optimal investment strategy problem of stock price with jump when risk is unhedged under mean-variance target criterion.

This also fully demonstrates the validity of the viscous solution theorem for dealing with random IQ problems with non-negative constraints. The results of this paper show that the short selling constraint of risk assets and the jump in the price process cannot be ignored on the effective strategy.

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